Generalized Kirchhoff approximation for Helmholtz equation

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GENERALIZED KIRCHHOFF APPROXIMATION FOR HELMHOLTZ EQUATION

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ABSTRACT. We give integral formulas to approximate solutions of Dirichlet and Neumann problems for Helmholtz equation at high frequencies. These approximations are valid in the complementary of a union of convex compact obstacles. The first step of the iterative procedure is the classical Kirchhoff approximation. Convergence is proved by comparison with the geometrical optics asymptotics. The method is shown to be numerically stable.

1. INTRODUCTION

Let Ω be an open set in $\mathbb{R}^3$ and $\Omega' = \mathbb{R}^3 \setminus \Omega$. We study the high frequency diffraction problems of an incident plane wave in Ω for Helmholtz equation with respectively, Dirichlet and Neumann boundary conditions :

$$(D) \qquad \begin{cases} \Delta v(x) + k^2 v(x) &= 0 \quad \text{for } x \in \Omega, \\ v(x) &= 0 \quad \text{for } x \in \partial\Omega, \\ v(x) &= e^{-ik\langle \xi, x\rangle} + u(x) \quad \text{for } x \in \Omega, \end{cases}$$

$$(N) \qquad \begin{cases} \Delta v(x) + k^2 v(x) &= 0 \quad \text{for } x \in \Omega, \\ \dfrac{\partial v}{\partial n}(x) &= 0 \quad \text{for } x \in \partial\Omega, \\ v(x) &= e^{-ik\langle \xi, x\rangle} + u(x) \quad \text{for } x \in \Omega. \end{cases}$$

Here u satisfies the Sommerfeld radiation condition :

$$r^2(\frac{\partial u}{\partial r} + iku) \text{ bounded when } r = \mid x \mid \to +\infty.$$

The incident plane wave, $e^{-ik\langle \xi, x\rangle}$, is given with the normalization $\mid \xi \mid = 1$. Here, *high frequency* means that the wave length is small with respect to $\partial\Omega$ curvatures. So usual numerical methods, such as finite element method, boundary element method and so on, fall down.

A classical high frequency approximation is given by geometrical optics , which, for a point $x \in \Omega$, allows us to compute, from optic rays going through x, an approximation of the diffracted wave by Ω'. We obtain for Dirichlet problem (D) and Neumann problem (N) respectively

$$v^D_{O.G.}(x) = \sum e^{-ik\varphi(x)} a^D_0(x) \tag{1.1}$$

and

$$v^N_{O.G.}(x) = \sum e^{-ik\varphi(x)} a^N_0(x). \tag{1.2}$$

Here, the phase $\varphi(x)$ is the length of optic rays going through x and computation of the amplitude $a^{\cdot}_0(x)$ work out by propagation and reflection formulas along optic rays (see [Cuv13]). The main problem of this method is its numerical instability : in order to compute this approximation, it's necessary to determine **all** the optic rays going through x. But, small errors in the numerical representation of $\partial\Omega$ can give large errors in the optic rays determination.

An other one is Kirchhoff approximation, based on integral representations. We give Kirchhoff approximation respectively for Dirichlet problem (D) and Neumann problem (N) :

$$v^D_{\text{Kir.}}(x) = e^{-ik\langle \boldsymbol{\xi},x\rangle} + \frac{1}{4\pi}\int_{\partial\Omega} ik\left(\langle \boldsymbol{\xi}, \boldsymbol{n}(\sigma)\rangle - |\langle \boldsymbol{\xi}, \boldsymbol{n}(\sigma)\rangle|\right) e^{-ik\langle \boldsymbol{\xi},\boldsymbol{\sigma}\rangle} \frac{e^{-ik|x-\sigma|}}{|\ x-\sigma\ |} d\sigma \tag{1.3}$$

$$v^N_{\text{Kir.}}(x) = e^{-ik\langle \boldsymbol{\xi},x\rangle} + \frac{1}{4\pi}\int_{\partial\Omega} ik\left(\frac{\langle \boldsymbol{\xi}, \boldsymbol{n}(\sigma)\rangle}{|\langle \boldsymbol{\xi}, \boldsymbol{n}(\sigma)\rangle|} - 1\right)\left\langle \frac{x-\sigma}{|x-\sigma|}, \boldsymbol{n}(\sigma)\right\rangle e^{-ik\langle \boldsymbol{\xi},\boldsymbol{\sigma}\rangle} \frac{e^{-ik|x-\sigma|}}{|x-\sigma|} d\sigma \tag{1.4}$$

Here, $\boldsymbol{n}(\sigma)$ is the unit normal to Γ at point σ, exterior to Ω'. But, the validity of this method is restricted to Ω' beeing a strictly convex compact (see [MT85] for Dirichlet problem and [Yin79], [Yin83] for Neumann problem) and false otherwise. This is due to the incapacity of this method to *see* multiple reflections.

The purpose of this paper is to determine an iterative integral method, numerically stable, equivalent, at first order and high frequency, to the geometrical optic approximation sets Ω' is a finite and disjointed union of regular and strictly convex compacts. In both case, the first step is given by Kirchhoff approximation.

It relies in ...(crire les principales tapes du papier)

2. Notations and definitions

2.1. Gradient and Hessian on Surfaces.

Let $K \subset \mathbb{R}^3$ be a compact and Γ it boundary. Let us suppose that Γ is a regular and orientable surface.

Definition 1. ***(Gradient on Surfaces)****. The* ***gradient*** *of a differentiable function $\varphi : \Gamma \subset \mathbb{R}^3 \to \mathbb{R}$ is a differentiable map $\boldsymbol{grad}\,\varphi : \Gamma \to \mathbb{R}^3$ which assigns to each point $\sigma \in \Gamma$ a vector $\boldsymbol{grad}\,\varphi(\sigma) \in T_\sigma(\Gamma) \subset \mathbb{R}^3$ such that*

$$< \boldsymbol{grad}\,\varphi(\sigma), v >_\sigma = d\varphi_\sigma(v) \quad \forall v \in T_\sigma(\Gamma).$$

Definition 2. ***(Hessian on Surfaces)****. The* ***hessian*** *of a twice differentiable function $\varphi : \Gamma \subset \mathbb{R}^3 \to \mathbb{R}$ is the function $\operatorname{Hess}\varphi : \Gamma \to \mathcal{L}(\mathbb{R}^{\nVdash})$ which assigns to each point $\sigma \in \Gamma$ a matrix $\operatorname{Hess}\varphi(\sigma) \in \mathcal{L}\left(T_\sigma(\Gamma)\right)$ such that*

$$\langle \operatorname{Hess}\varphi(\sigma)v, v\rangle = d^2\varphi_\sigma(v).v \quad \forall v \in T_\sigma(\Gamma).$$

Proposition 1. *With previous definition, and by taylor's expansion we obtain for $t \in \mathbb{R}$ and $v \in T_\sigma(\Gamma)$*

$$\varphi(\sigma + tv) - \varphi(\sigma) = t\,\langle \boldsymbol{grad}\,\varphi(\sigma), v\rangle + \frac{t^2}{2}\langle \operatorname{Hess}\varphi(\sigma)v, v\rangle + o(t^2)$$

Definition 3. *The gradient of a differentiable function $\varphi : \overbrace{\Gamma \times \cdots \times \Gamma}^{N \text{ times}} \subset \mathbb{R}^{3N} \to \mathbb{R}$ is a differentiable map $\boldsymbol{grad}\,\varphi : \Gamma \times \cdots \times \Gamma \to \mathbb{R}^{3N}$ which assigns to each point $\nu = (\sigma_1, \ldots, \sigma_N) \in \Gamma^N$ a vector $\boldsymbol{grad}\,\varphi(\nu) \in (T_{\sigma_1}(\Gamma) \times \cdots \times T_{\sigma_N}(\Gamma)) \subset \mathbb{R}^{3N}$ such that*

$$\langle \boldsymbol{grad}\,\varphi(\nu), \varpi\rangle_\nu = d\varphi_\nu(\varpi) \quad \forall \varpi \in (T_{\sigma_1}(\Gamma) \times \cdots \times T_{\sigma_N}(\Gamma))$$

The function $\boldsymbol{\nabla}_{\sigma_i}\varphi : \Gamma^N \to \mathbb{R}^3$ which assigns to each point $\nu = (\sigma_1, \ldots, \sigma_N) \in \Gamma^N$ a vector $\boldsymbol{\nabla}_{\sigma_i}\varphi(\nu) \in T_{\sigma_i}(\Gamma) \subset \mathbb{R}^3$ is defined by

$$\langle \boldsymbol{\nabla}_{\sigma_i}\varphi(\nu), \varpi_i\rangle = d\varphi_\nu(\varpi) \quad \forall \varpi_i \in T_{\sigma_i}(\Gamma)$$

with $\varpi = (0, \ldots, 0, \varpi_i, 0, \ldots, 0)$.

Definition 4. *The* ***hessian*** *of a twice differentiable function* $\varphi : \Gamma^N \to \mathbb{R}$ *is the function* $\mathrm{Hess}\,\varphi : \overbrace{\Gamma \times \cdots \times \Gamma}^{N \text{ times}} \to \mathcal{L}\left(\mathbb{R}^{3N}\right)$ *which assigns to each point* $\nu = (\sigma_1, \ldots, \sigma_N) \in \Gamma^N$, *a matrix* $\mathrm{Hess}\,\varphi(\nu) \in \mathcal{L}\left(T_{\sigma_1}(\Gamma) \times \cdots \times T_{\sigma_N}(\Gamma)\right)$ *such that*

$$\langle \mathrm{Hess}\,\varphi(\nu)\varpi, \varpi \rangle_\nu = d^2\varphi_\nu(\varpi).\varpi \quad \forall \varpi \in T_{\sigma_1}(\Gamma) \times \cdots \times T_{\sigma_N}(\Gamma).$$

The function $\mathcal{H}_{i,j}\varphi : \overbrace{\Gamma \times \cdots \times \Gamma}^{N \text{ times}} \to \mathcal{L}\left(\mathbb{R}^3\right)$, $1 \leq i \neq j \leq N$, *which assigns to each point* $\nu = (\sigma_1, \ldots, \sigma_N) \in \Gamma^N$, *a matrix* $\mathcal{H}_{i,j}\varphi(\nu) \in \mathcal{L}\left(T_{\sigma_i}(\Gamma), T_{\sigma_j}(\Gamma)\right)$ *is defined by*

$$\langle \mathcal{H}_{i,j}\varphi(\nu)\omega_i, \omega_j \rangle = d^2\varphi_\nu(\varpi).\varpi \quad \forall (\omega_i, \omega_j) \in T_{\sigma_i}(\Gamma) \times T_{\sigma_j}(\Gamma)$$

with $\varpi = (\varpi_1, \ldots, \varpi_N)$, $\varpi_k = 0$ *for* $k \neq i$ *and* $k \neq j$, $\varpi_i = \omega_i$ *and* $\varpi_j = \omega_j$.

The function $\mathcal{H}_{i,i}\varphi : \overbrace{\Gamma \times \cdots \times \Gamma}^{N \text{ times}} \to \mathcal{L}\left(\mathbb{R}^3\right)$, $1 \leq i \leq N$, *which assigns to each point* $\nu = (\sigma_1, \ldots, \sigma_N) \in \Gamma^N$, *a matrix* $\mathcal{H}_{i,i}\varphi(\nu) \in \mathcal{L}\left(T_{\sigma_i}(\Gamma)\right)$ *is defined by*

$$\langle \mathcal{H}_{i,i}\varphi(\nu)\omega_i, \omega_i \rangle = d^2\varphi_\nu(\varpi).\varpi \quad \forall \omega_i \in T_{\sigma_i}(\Gamma)$$

with $\varpi = (\varpi_1, \ldots, \varpi_N) \in T_{\sigma_1}(\Gamma) \times \cdots \times T_{\sigma_N}(\Gamma)$, $\varpi_k = 0$ *for* $k \neq i$, *and* $\varpi_i = \omega_i$.

2.2. **Geometrical notations.**

- Let $(K_i)_{i=1,\cdots,N}$ be a set of regular, disjoint and strictly convex compact in $\mathbb{R}^3$.
- We denote by Γ_i, the boundary of K_i, and $\Gamma = \bigcup_{i=1}^N \Gamma_i$.
- Thus Γ_i is a regular and orientable surface. So, given a point σ of surface Γ_i we can choose the coordinate axis of $\mathbb{R}^3$ so that origin O of the coordinates is at σ and the z axis is directed along the **negative normal** (i.e. the outer normal) $\boldsymbol{n}(\sigma)$ of Γ_i in σ (thus, the xy plane agrees with $T_\sigma(\Gamma_i)$: tangent plane of Γ_i in σ). It follows that a neighborhooh of σ in Γ_i can be represented in the form $z = g_i(u, v)$, $(u, v) \in U \subset \mathbb{R}^2$, where U is an open set and g_i is a differentiable function with $g_i(0,0) = \frac{\partial g_i}{\partial u}(0,0) = \frac{\partial g_i}{\partial v}(0,0) = 0$.

  Let us assume further that the u and v axes are directed along the **principal directions**, with the axis u along the direction of maximum principal curvature. Thus

  $$k_1^i(\sigma) = \frac{\partial^2 g_i}{\partial u^2}(0,0), \; k_2^i(\sigma) = \frac{\partial^2 g_i}{\partial v^2}(0,0), \quad \text{and } \frac{\partial^2 g_i}{\partial u \partial v}(0,0) = 0$$

  and, so we obtain by developing $g_i(u, v)$ into Taylor's expansion about $(0,0)$

  $$g_i(u, v) = -\frac{1}{2}(k_1^i u^2 + k_2^i v^2) + o(u^2 + v^2)$$

- We note $R_u^i(\sigma) = \frac{1}{k_1^i(\sigma)}$ and $R_v^i(\sigma) = \frac{1}{k_2^i(\sigma)}$ the **principal radii of curvature**.
- Let us denote $\Re_i(\sigma)$ the orthonormal basis $\Re_i(\sigma) = \{\boldsymbol{u}_i, \boldsymbol{v}_i, \boldsymbol{n}_i\}$ where $\boldsymbol{n}_i$ is the negative normal of Γ_i in σ, $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ are the principal directions of Γ_i in σ with $\boldsymbol{u}_i$ the direction of maximum principal curvature.
- We set $\Omega' = \bigcup\limits_{i=1}^N K_i$ and we note $\Omega_i = \mathbb{R}^3 \setminus K_i$.
- Let $\mathcal{M}_{m,n}(\mathbb{R})$ the set of real matrix of size $m \times n$.

- We note

$$\Gamma_*^l = \left\{(\sigma_1, \cdots, \sigma_l) \in \Gamma^l \text{ with } (\sigma_j \in \Gamma_p \text{ and } \sigma_{j+1} \in \Gamma_q \Rightarrow p \neq q)\right\}$$

and the phase function $\psi_l : \mathbb{R}^3 \times \Gamma_*^l \to \mathbb{R}$ defined, for all $\nu = \sigma_1, \cdots, \sigma_l \in \Gamma_*^l$, by

$$\psi_l(x;\nu) = \langle \boldsymbol{\xi}, \sigma_1 \rangle + \sum_{j=1}^{l-1} |\sigma_{j+1} - \sigma_j| + |x - \sigma_l|$$

- We call $\mathcal{C}_l(x)$ the set of all l-uplet $(\sigma_1, \cdots, \sigma_l) \in \Gamma_*^l$ such that:
  (1) $\langle \boldsymbol{\xi}, \boldsymbol{n}(\sigma_1) \rangle < 0$ and $\langle \sigma_{j+1} - \sigma_j, \boldsymbol{n}(\sigma_{j+1}) \rangle < 0$ for all $j \in \{1, \cdots, l-1\}$,
  (2) The phase $\psi_l(x, \bullet)$ is stationary on Γ_*^l at point $\nu = (\sigma_1, \cdots, \sigma_l)$ (i.e. $\boldsymbol{grad}_\nu\, \psi_l(x;\nu) = 0$)

  We note $\mathcal{C}(x) = \bigcup_l \mathcal{C}_l(x)$.
- Let $\nu = (\sigma_1^\nu, \cdots, \sigma_l^\nu) \in \mathcal{C}_l(x)$. We note $\Re_j^\nu = \Re_j(\sigma_j^\nu) = \{\boldsymbol{u}_j^\nu, \boldsymbol{v}_j^\nu, \boldsymbol{n}_j^\nu\}$, $\quad j \in \{1, \cdots, l\}$.
- Let $\nu = (\sigma_1^\nu, \cdots, \sigma_l^\nu) \in \mathcal{C}_l(x)$, we define $\boldsymbol{\xi}_j^\nu$, for $j \in \{1, \cdots, l\}$ by :

$$\sigma_{j+1}^\nu - \sigma_j^\nu = \lambda_j^\nu \boldsymbol{\xi}_j^\nu \text{ with } \lambda_j^\nu = \mid \sigma_{j+1}^\nu - \sigma_j^\nu \mid$$

and note $\boldsymbol{\xi}_j^\nu = \left(\xi_{j,1}^\nu, \xi_{j,2}^\nu, \xi_{j,3}^\nu\right)_{\Re_j^\nu}$. We also note $\mathbb{B}(\sigma_j^\nu)$ the curvature matrix of Γ_j in σ_j^ν : it's the diagonal matrix with diagonal entries $\left(\frac{1}{U_j^\nu}, \frac{1}{V_j^\nu}, 0\right)$ in $\Re_j^\nu$ where $U_j^\nu = 1/k_1^j(\sigma_j^\nu)$ and $V_j^\nu = 1/k_2^j(\sigma_j^\nu)$ are the principal radius of curvature. Due to strict convexity of compact we have $U_j^\nu > 0$ and $V_j^\nu > 0$.
- Let $\mathbb{R}_j^\nu$ defined by :

$$\mathbb{R}_j^\nu = \begin{pmatrix} \langle \boldsymbol{u}_{j+1}^\nu, \boldsymbol{u}_j^\nu \rangle & \langle \boldsymbol{u}_{j+1}^\nu, \boldsymbol{v}_j^\nu \rangle & \langle \boldsymbol{u}_{j+1}^\nu, \boldsymbol{n}_j^\nu \rangle \\ \langle \boldsymbol{v}_{j+1}^\nu, \boldsymbol{u}_j^\nu \rangle & \langle \boldsymbol{v}_{j+1}^\nu, \boldsymbol{v}_j^\nu \rangle & \langle \boldsymbol{v}_{j+1}^\nu, \boldsymbol{n}_j^\nu \rangle \\ \langle \boldsymbol{n}_{j+1}^\nu, \boldsymbol{u}_j^\nu \rangle & \langle \boldsymbol{n}_{j+1}^\nu, \boldsymbol{v}_j^\nu \rangle & \langle \boldsymbol{n}_{j+1}^\nu, \boldsymbol{n}_j^\nu \rangle \end{pmatrix}$$

- we call $\mathcal{R}_l(x)$ the set of all l-uplet $\rho = (\sigma_1, \cdots, \sigma_l) \in (\partial\Omega)_*^l$ such that ρ is an optic ray going through x and σ_i the point of i^{th} reflection along this ray. We note $\mathcal{R}(x) = \bigcup_l \mathcal{R}_l(x)$.
- Let $x \in \Omega$ and $\rho = (\sigma_1, \cdots, \sigma_l) \in \mathcal{C}_l(x)$. We say that ρ realize :
  (1) *a transmission condition at point* σ_i $(i = 1, \cdots, l)$ if

$$\begin{cases} \frac{\sigma_2 - \sigma_1}{|\sigma_2 - \sigma_1|} & = & \boldsymbol{\xi} & \text{for } i = 1 \\ \frac{\sigma_{i+1} - \sigma_i}{|\sigma_{i+1} - \sigma_i|} & = & \frac{\sigma_i - \sigma_{i-1}}{|\sigma_i - \sigma_{i-1}|} & \text{for } i \in \{2, \cdots, l\} \end{cases}$$

  (2) *a reflection condition at point* σ_i $(i = 1, \cdots, l)$ if

$$\begin{cases} \frac{\sigma_2 - \sigma_1}{|\sigma_2 - \sigma_1|} & = & \boldsymbol{\xi} - 2\langle \boldsymbol{\xi}, \boldsymbol{n}(\sigma_1) \rangle \boldsymbol{n}(\sigma_1) \quad \text{for } i = 1 \\ \frac{\sigma_{i+1} - \sigma_i}{|\sigma_{i+1} - \sigma_i|} & = & \frac{\sigma_i - \sigma_{i-1}}{|\sigma_i - \sigma_{i-1}|} - 2\left\langle \frac{\sigma_i - \sigma_{i-1}}{|\sigma_i - \sigma_{i-1}|}, \boldsymbol{n}(\sigma_i) \right\rangle \boldsymbol{n}(\sigma_i) \;\; \text{for } i \in \{2, \cdots, l\} \end{cases}$$

  Here $\sigma_{l+1} = x$.
- Let $x \in \Omega$ and $\nu = (\sigma_1^\nu, \cdots, \sigma_l^\nu) \in \mathcal{C}_l(x)$ we note

$$\delta^\nu(\sigma_j^\nu) = \begin{cases} 0 \text{ if } \sigma_j^\nu \text{ is a transmission point} \\ 1 \text{ if } \sigma_j^\nu \text{ is a reflexion point} \end{cases}$$

- We note $\mathcal{T} = \bigcup_l \mathcal{T}_l$ with $\mathcal{T}_l$ the set of points $x \in \Omega$ such that exists $(\sigma_1, \cdots, \sigma_l) \in (\partial\Omega)_*^l$ verifying
  (1) $\langle \boldsymbol{\xi}, \boldsymbol{n}(\sigma_1) \rangle = 0$ **or** $\exists j \in \{1, \cdots, l-1\}$ such that $\langle \sigma_{j+1} - \sigma_j, \boldsymbol{n}(\sigma_{j+1}) \rangle = 0$,
  (2) The phase $\psi_l(x, \bullet)$ is stationary on $(\partial\Omega)_*^l$ in $(\sigma_1, \cdots, \sigma_l)$.

2.3. **Matrix applications.**

- Let $\sigma > 0$, we note $\mathcal{S}_\sigma \subset \mathcal{M}_{3,3}(\mathbb{R})$ the set of matrix $\mathbb{A}$ such that $\mathbb{I} + \sigma\mathbb{A}$ is regular. We note S_σ the following application :

$$S_\sigma \quad : \quad \begin{array}{rcl} \mathcal{S}_\sigma & \longrightarrow & \mathcal{M}_{3,3}(\mathbb{R}) \\ \mathbb{A} & \longmapsto & \mathbb{A}(\mathbb{I} + \sigma\mathbb{A})^{-1} \end{array}$$

- Let $\mathbb{B} \in \mathcal{M}_{3,3}(\mathbb{R})$ a symmetric matrix , $\boldsymbol{\eta} \in \mathbb{R}^3$, $\boldsymbol{\zeta} \in \mathbb{R}^3$. We suppose $\langle \boldsymbol{\zeta}, \eta \rangle \neq 0$. We note $T_{\mathbb{B},\boldsymbol{\eta},\boldsymbol{\zeta}}$ the application of $\mathcal{M}_{3,3}(\mathbb{R})$ given by :
$\forall \mathbb{A} \in \mathcal{M}_{3,3}(\mathbb{R}),\ \forall x \in \mathbb{R}^3$

$$\begin{aligned}(T_{\mathbb{B},\boldsymbol{\eta},\boldsymbol{\zeta}}(\mathbb{A}))x \quad = \quad & (\mathbb{A} - 2\left\langle \boldsymbol{\zeta}, \boldsymbol{\eta} \right\rangle \mathbb{B})x - 2\left\langle \boldsymbol{\eta}, x \right\rangle (\mathbb{A}\boldsymbol{\eta} + \mathbb{B}\boldsymbol{\zeta}) \\ & -2\left\langle \mathbb{A}\boldsymbol{\eta} + \mathbb{B}\boldsymbol{\zeta}, x \right\rangle \boldsymbol{\eta} + 2\left[2\left\langle \mathbb{A}\boldsymbol{\eta}, \boldsymbol{\eta} \right\rangle - \frac{\langle \mathbb{B}\boldsymbol{\zeta}, \boldsymbol{\zeta} \rangle}{\langle \boldsymbol{\zeta}, \boldsymbol{\eta} \rangle} \right] \left\langle \boldsymbol{\eta}, x \right\rangle \boldsymbol{\eta}\end{aligned}$$

- we define, for $x \in \Omega \setminus \mathcal{T}$ and $\nu = (\sigma_1^\nu, \cdots, \sigma_l^\nu) \in \mathcal{C}_l(x)$, the following l matrices $\mathbb{M}_j^\nu$ in $\mathcal{M}_{2,2}(\mathbb{R})$:

$$\mathbb{M}_1^\nu = \overline{\mathcal{H}}_{1,1}\psi_l(x, \nu)$$

and, $\forall j \in \{2, \cdots, l\}$

$$\mathbb{M}_j^\nu = \overline{\mathcal{H}}_{j,j}\psi_l(x;\nu) - \overline{\mathcal{H}}_{j-1,j}\psi_l(x;\nu) \left[\mathbb{M}_{j-1}^\nu\right]^{-1} \overline{\mathcal{H}}_{j,j-1}\psi_l(x;\nu).$$

Remark 1. *We shall see in Lemma 3 that $\mathbb{M}_j^\nu$ is regular.*

- Let $x \in \Omega \setminus \mathcal{T}$ and $\nu = (\sigma_1^\nu, \cdots, \sigma_l^\nu) \in \mathcal{C}_l(x)$. We define by recurrence the l symmetric matrices $\mathbb{P}_j^\nu$ in $\mathcal{L}(\mathbb{R}^3)$ such that

$$\mathbb{P}_1^\nu = T_{\mathbb{B}(\sigma_1^\nu), n(\sigma_1^\nu), \xi}\,(0) \times \delta^\nu(\sigma_1^\nu)$$

and, $\forall j \in \{2, \cdots, l\}$

$$\mathbb{P}_j^\nu = S_{\lambda_{j-1}^\nu}(\mathbb{P}_{j-1}^\nu) \times \left(1 - \delta^\nu(\sigma_j^\nu)\right) + T_{\mathbb{B}(\sigma_j^\nu), n(\sigma_j^\nu), \xi_{j-1}^\nu}\left(S_{\lambda_{j-1}^\nu}(\mathbb{P}_{j-1}^\nu)\right) \times \delta^\nu(\sigma_j^\nu).$$

- Let $\mathcal{D}_l$ the function define on Γ_*^l by

$$\mathcal{D}_l(\sigma_1, \cdots, \sigma_l) \\ = \\ \left(|\langle \boldsymbol{\xi}, \boldsymbol{n}(\sigma_1) \rangle| - \langle \boldsymbol{\xi}, \boldsymbol{n}(\sigma_1) \rangle\right) \prod_{j=1}^{l-1} \frac{\left[\left|\left\langle \frac{\sigma_{j+1} - \sigma_j}{|\sigma_{j+1} - \sigma_j|}, \boldsymbol{n}(\sigma_{j+1}) \right\rangle\right| - \left\langle \frac{\sigma_{j+1} - \sigma_j}{|\sigma_{j+1} - \sigma_j|}, \boldsymbol{n}(\sigma_{j+1}) \right\rangle\right]}{|\sigma_{j+1} - \sigma_j|}$$

- Let $\mathcal{N}_l$ the function define on Γ_*^l by

$$\mathcal{N}_l(\sigma_1, \cdots, \sigma_l) \\ = \\ \left(\frac{\langle \boldsymbol{\xi}, \boldsymbol{n}(\sigma_1) \rangle}{|\langle \boldsymbol{\xi}, \boldsymbol{n}(\sigma_1) \rangle|} - 1\right) \prod_{j=1}^{l-1} \left[\frac{\left\langle \frac{\sigma_{j+1} - \sigma_j}{|\sigma_{j+1} - \sigma_j|}, \boldsymbol{n}(\sigma_{j+1}) \right\rangle}{\left|\left\langle \frac{\sigma_{j+1} - \sigma_j}{|\sigma_{j+1} - \sigma_j|}, \boldsymbol{n}(\sigma_{j+1}) \right\rangle\right|} - 1 \right] \left\langle \frac{\sigma_{j+1} - \sigma_j}{|\sigma_{j+1} - \sigma_j|}, \boldsymbol{n}(\sigma_j) \right\rangle$$

3. Geometrical optics approximation

We only give the main results. The geometrical optic approximation is given, for Dirichlet and Neumann problems, respectively by : $\forall x \in \Omega \setminus \mathcal{T}$

$$v_{O.G.}^D(x) \\ = \\ e^{-ik\langle \boldsymbol{\xi}, x \rangle} + \sum_{l \geq 1} (-1)^l \sum_{\rho = (\sigma_1^\rho, \cdots, \sigma_l^\rho) \in \mathcal{R}_l(x)} \frac{e^{-ik\psi_l(x;\rho)}}{\prod_{j=1}^{l} \sqrt{\det(\mathbb{I} + |\sigma_{j+1}^\rho - \sigma_j^\rho|\mathbb{P}_j^\rho)}} \tag{3.1}$$

and

$$v^N_{O.G.}(x) = e^{-ik\langle\boldsymbol{\xi},x\rangle} + \sum_{l\geq 1}\sum_{\rho=(\sigma_1^\rho,\cdots,\sigma_l^\rho)\in\mathcal{R}_l(x)} \frac{e^{-ik\psi_l(x;\rho)}}{\prod_{j=1}^{l}\sqrt{\det(\mathbb{I}+|\sigma^\rho_{j+1}-\sigma^\rho_j|\mathbb{P}^\rho_j)}} \tag{3.2}$$

Here $\sigma_{l+1} = x$, $\det\left(I+ \mid \sigma^\rho_{j+1} - \sigma^\rho_j \mid P^\rho_j\right)$ is positive, and in corollary, the Maslov indice vanished. For more explanation, report to [eC89] or [Cuv13].

4. Iterative Kirchhoff approximation

4.1. **Dirichlet problem.** We introduce the following kernels series

$$p^D_1(\sigma) = ik\left(|\langle\xi, n(\sigma)\rangle| - \langle\xi, n(\sigma)\rangle\right)e^{-ik\langle\xi,\sigma\rangle} \quad \forall\sigma\in\partial\Omega$$

and, for $\sigma\in\partial K_j$

$$p^D_l(\sigma) = \frac{ik}{4\pi}\int_{\partial\Omega\setminus\partial K_j} p^D_{l-1}(\sigma')\left[\left|\frac{\sigma-\sigma'}{\mid\sigma-\sigma'\mid}.n(\sigma)\right| - \frac{\sigma-\sigma'}{\mid\sigma-\sigma'\mid}.n(\sigma)\right]\frac{e^{-ik|\sigma'-\sigma|}}{\mid\sigma'-\sigma\mid}d\sigma'$$

We set the **iterative Kirchhoff approximation** for Dirichlet problem (D) by

$$\left\{\begin{array}{lll} u^D_0(x) & = & 0\\ u^D_l(x) & = & u^D_{l-1}(x) + \frac{1}{4\pi}\int_{\partial\Omega} p^D_l(\sigma)\frac{e^{-ik|x-\sigma|}}{|x-\sigma|}d\sigma \end{array}\right. \tag{4.1}$$

That is to say with previous notations

$$u^D_l(x) = u^D_{l-1}(x) + \left(\frac{ik}{4\pi}\right)^l\int_{(\partial\Omega)^l_*}\mathcal{D}_l(\sigma_1,\cdots,\sigma_l)\frac{e^{-ik\psi_l(x;\sigma_1,\cdots,\sigma_l)}}{\mid x-\sigma_l\mid}d\sigma_1\cdots d\sigma_l \tag{4.2}$$

We state the main result comparing the iterative method describe in (4.1) and the geometrical optic approximation given in (3.1) for problem (D) :

Theorem 1. *Let Ω an open in $\mathbb{R}^3$, exterior of a regular domain Ω' finite and disjointed reunion of strictly convex compacts. Let $x\in\Omega\setminus\mathcal{T}$. If $\mathcal{C}_l(x)=\emptyset$ for $l>n$ then*

$$e^{-ik\langle\xi,x\rangle} + u^D_n(x) - v^D_{0.G.}(x) = O\left(\frac{1}{k}\right) \tag{4.3}$$

locally uniformly in x.

4.2. **Neumann problem.** We introduce the following kernels series

$$p^N_1(\sigma) = ik\left(\frac{\langle\xi,n(\sigma)\rangle}{|\langle\xi,n(\sigma)\rangle|} - 1\right)e^{-ik\langle\xi,\sigma\rangle} \quad \forall\sigma\in\partial\Omega$$

and, for $\sigma\in\partial K_j$

$$\begin{array}{rcl} p^N_l(\sigma) & = & \displaystyle\frac{ik}{4\pi}\int_{\partial\Omega\setminus\partial K_j} p^N_{l-1}(\sigma')\left[\frac{\left\langle\frac{\sigma-\sigma'}{|\sigma-\sigma'|}, n(\sigma)\right\rangle}{\left|\left\langle\frac{\sigma-\sigma'}{|\sigma-\sigma'|}, n(\sigma)\right\rangle\right|} - 1\right]\\ & & \displaystyle\times\left\langle\frac{\sigma-\sigma'}{\mid\sigma-\sigma'\mid}, n(\sigma')\right\rangle\frac{e^{-ik|\sigma'-\sigma|}}{\mid\sigma'-\sigma\mid}d\sigma' \end{array}$$

We set the **iterative Kirchhoff approximation** for Neumann problem (N) by

$$\left\{\begin{array}{lll} u^N_0(x) & = & 0\\ u^N_l(x) & = & u^N_{l-1}(x) + \frac{1}{4\pi}\int_{\partial\Omega} p^N_l(\sigma)\left\langle\frac{x-\sigma}{|x-\sigma|}, n(\sigma)\right\rangle\frac{e^{-ik|x-\sigma|}}{|x-\sigma|}d\sigma \end{array}\right. \tag{4.4}$$

That is to say with previous notations

$$u_l^N(x) - u_{l-1}^N(x) \\ = \\ \left(\frac{ik}{4\pi}\right)^l \int_{(\partial\Omega)_*^l} \mathcal{N}_l(\sigma_1, \cdots, \sigma_l) \left\langle \frac{x-\sigma_l}{|x-\sigma_l|}, n(\sigma_l) \right\rangle \frac{e^{-ik\psi_l(x;\sigma_1,\cdots,\sigma_l)}}{|x-\sigma_l|} d\sigma_1 \cdots d\sigma_l \tag{4.5}$$

We state the main result comparing the iterative method describe in (4.4) and the geometrical optic approximation given in (3.2) for problem (N) :

Theorem 2. *Let Ω an open in $\mathbb{R}^3$, exterior of a regular domain Ω' finite and disjointed reunion of strictly convex compacts. Let $x \in \Omega \setminus \mathcal{T}$. If $\mathcal{C}_l(x) = \emptyset$ for $l > n$ then*

$$e^{-ik\langle \xi, x\rangle} + u_n^N(x) - v_{0.G.}^N(x) = O\left(\frac{1}{k}\right) \tag{4.6}$$

locally uniformly in x.

5. Technical Lemmas and properties

To prove previous theorems we need some technical lemmas and properties.

5.1. Stationary phase points of $\psi_l(x;\bullet)$.

We first remark that

Remark 2. *Let $\nu = (\sigma_1^\nu, \cdots, \sigma_l^\nu) \in (\partial\Omega)_*^l$, if*

$$\langle \xi, n(\sigma_1^\nu)\rangle \geq 0$$

<u>or</u> if exists $j \in \{1, \cdots, l-1\}$ such that

$$\left\langle \frac{\sigma_{j+1}^\nu - \sigma_j^\nu}{|\sigma_{j+1}^\nu - \sigma_j^\nu|}, n(\sigma_{j+1}^\nu) \right\rangle \geq 0$$

then

$$\mathcal{D}_l(\sigma_1^\nu, \cdots, \sigma_l^\nu) = \mathcal{N}_l(\sigma_1^\nu, \cdots, \sigma_l^\nu) = 0$$

To find stationary phase points on $(\partial\Omega)_*^l$, we have to compute, for all $x \in \mathbb{R}^3$, the set of points $\nu = (\sigma_1^\nu, \cdots, \sigma_l^\nu) \in (\partial\Omega)_*^l$ which satisfies

$$\nabla_\nu^S \psi_l(x; \sigma_1^\nu, \cdots, \sigma_l^\nu) = 0 \tag{5.1}$$

We have

$$\nabla_\nu \psi_l(x; \sigma_1^\nu, \cdots, \sigma_l^\nu) = \begin{pmatrix} \xi - \frac{\sigma_2^\nu - \sigma_1^\nu}{|\sigma_2 - \sigma_1|} \\ \frac{\sigma_2^\nu - \sigma_1^\nu}{|\sigma_2^\nu - \sigma_1^\nu|} - \frac{\sigma_3^\nu - \sigma_2^\nu}{|\sigma_3^\nu - \sigma_2^\nu|} \\ \vdots \\ \frac{\sigma_l^\nu - \sigma_{l-1}^\nu}{|\sigma_l^\nu - \sigma_{l-1}^\nu|} - \frac{x - \sigma_l^\nu}{|x - \sigma_l^\nu|} \end{pmatrix}$$

The condition (5.1) is equivalent to the existence of $(\mu_1, \cdots, \mu_l) \in \mathbb{R}^l$ such that

$$\nabla_\nu \psi_l(x; \sigma_1^\nu, \cdots, \sigma_l^\nu) = \begin{pmatrix} \mu_1 n(\sigma_1^\nu) \\ \vdots \\ \mu_l n(\sigma_l^\nu) \end{pmatrix}$$

By hypothesis $|\xi| = 1$, so we obtain

$$\left\{ \begin{array}{ll} & \mu_1 = 0 \quad \text{or} \quad \mu_1 = 2\,\langle \xi, n(\sigma_1^\nu)\rangle\, n(\sigma_1^\nu) \\ \text{and} \quad \forall j \in \{2, \cdots, l\} & \\ & \mu_j = 0 \quad \text{or} \quad \mu_j = 2\left\langle \frac{\sigma_j^\nu - \sigma_{j-1}^\nu}{|\sigma_j^\nu - \sigma_{j-1}^\nu|}, n(\sigma_j^\nu) \right\rangle n(\sigma_j^\nu) \end{array} \right.$$

Then, using remark 2, we have the

Lemma 1. *Let $x \in \Omega \setminus \mathcal{T}$ and $\nu = (\sigma_1^\nu, \cdots, \sigma_l^\nu) \in \mathcal{C}_l(x)$. Then ν realize a transmission or a reflection condition on each points σ_j^ν, $j \in \{1, \ldots, l\}$.*

5.2. **Relation between $\mathbb{M}_j^\nu$ and $\mathbb{P}_j^\nu$.** We have the fundamental Lemma

Lemma 2. *Let $x \in \Omega \setminus \mathcal{T}$ and $\nu = (\sigma_1^\nu, \cdots, \sigma_l^\nu) \in C_l(x)$. Then, $\forall j \in \{1, \cdots, l\}$ we have :*

$$\mathbb{M}_j^\nu = \mathbb{P}^\nu_{j\,|T_{\partial\Omega}(\sigma_j^\nu)} + \frac{1}{\lambda_j^\nu}\left(\mathbb{I} - (\xi_j^\nu\ {}^t\xi_j^\nu)_{|T_{\partial\Omega}(\sigma_j^\nu)}\right), \tag{5.2}$$

$$\det \mathbb{M}_j^\nu = \left(\frac{\langle \xi_j^\nu, n(\sigma_j^\nu)\rangle}{\lambda_j^\nu}\right)^2 \det(\mathbb{I} + \lambda_j^\nu \mathbb{P}_j^\nu), \tag{5.3}$$

$$\operatorname{sgn} \mathbb{M}_j^\nu = 2, \tag{5.4}$$

and

$$\mathbb{M}_j^\nu \quad \textit{inversible.} \tag{5.5}$$

Proof of lemma 2 :
The proof of the lemma worked out by recurrence.

step one of recurrence proof

Using definition of $\mathbb{P}_1^\nu$, we easily obtain

$$\mathbb{P}^\nu_{1\,|T_{\partial\Omega}(\sigma_1^\nu)} = 2\,\langle \xi_1^\nu, n(\sigma_1^\nu)\rangle\, \delta^\nu(\sigma_1^\nu)\mathbb{B}(\sigma_1^\nu)_{|T_{\partial\Omega}(\sigma_1^\nu)}$$

On the other hand, we have

$$\begin{array}{rcl} \mathbb{M}_1^\nu &=& H^S_{\sigma_1,\sigma_1}\psi_l(x;\nu) \\ &=& H^S_{\sigma_1,\sigma_1}\left(\langle \xi, \sigma_1^\nu\rangle + |\sigma_2^\nu - \sigma_1^\nu|\right) \\ &=& 2\,\langle \xi_1^\nu, n(\sigma_1^\nu)\rangle\, \delta^\nu(\sigma_1^\nu)\mathbb{B}(\sigma_1^\nu)_{|T_{\partial\Omega}(\sigma_1^\nu)} + \frac{1}{\lambda_1^\nu}\left(\mathbb{I} - (\xi_1^\nu\ {}^t\xi_1^\nu)_{|T_{\partial\Omega}(\sigma_1^\nu)}\right) \end{array}$$

and so we have proved formula (5.2) for $j = 1$.

To obtain formula (5.3), we first remark that $\mathbb{P}_1^\nu \xi_1^\nu = 0$. We set

$$\mathbb{H}_j = \begin{bmatrix} 1 & 0 & \xi^\nu_{j,1} \\ 0 & 1 & \xi^\nu_{j,2} \\ 0 & 0 & \xi^\nu_{j,3} \end{bmatrix}$$

Then, combining $\mid \xi_1^\nu \mid = 1$ and formula (5.2) gives

$$\begin{array}{rcl} \det(\mathbb{I} + \lambda_1^\nu \mathbb{P}_1^\nu) &=& \frac{1}{(\det \mathbb{H}_1)^2} \det\left(\ {}^t\mathbb{H}_1\,(\mathbb{I} + \lambda_1^\nu \mathbb{P}_1^\nu)\,\mathbb{H}_1\right) \\ &=& \left(\frac{\lambda_1^\nu}{\langle \xi_1^\nu, n(\sigma_1^\nu)\rangle}\right)^2 \det \mathbb{M}_1^\nu. \end{array}$$

One finds easily that $\det(\mathbb{I} + \lambda_1^\nu \mathbb{P}_1^\nu) > 0$ and $\operatorname{tr} \mathbb{M}_1^\nu > 0$. So, we get $\operatorname{sgn} \mathbb{M}_1^\nu = 2$. We deduce that $\det \mathbb{M}_1^\nu > 0$ and so $\mathbb{M}_1^\nu$ is regular.

Step $j + 1$ of recurrence proof

To prove formula (5.2) at step $j + 1$, we first have to compute $(\mathbb{P}^\nu_{j+1})_{\Re_{j+1}}$ and $\mathbb{M}^\nu_{j+1}$ respectively in function of $(\mathbb{P}_j^\nu)_{\Re_j^\nu}$ and $\mathbb{M}_j^\nu$. To simplify notations, we note $(\mathbb{P}_j^\nu)_{\Re_j^\nu} = [P_{r,s}]_{r,s\in\{1,2,3\}}$, and $\mathbb{M}_j^\nu = [M_{r,s}]_{r,s\in\{1,2\}}$.

- **Computation of** $(\mathbb{P}^\nu_{j+1})_{\Re_{j+1}}$
  By definition,

$$(\mathbb{P}^\nu_{j+1})_{\Re^\nu_{j+1}} = \left(S_{\lambda_j^\nu}(\mathbb{P}_j^\nu)\right)_{\Re^\nu_{j+1}} \times \left(1 - \delta^\nu(\sigma^\nu_{j+1})\right) + T_{\mathbb{B}(\sigma^\nu_{j+1}), n(\sigma^\nu_{j+1}), \xi_j^\nu}\left(S_{\lambda_j^\nu}(\mathbb{P}_j^\nu)\right)_{\Re^\nu_{j+1}} \times \delta^\nu(\sigma^\nu_{j+1}).$$

We first evaluate $S_{\lambda_j^\nu}(\mathbb{P}_j^\nu)_{\Re_j}$:

$$(S_{\lambda_j^\nu}(\mathbb{P}_j^\nu))_{\Re_j}\det(\mathbb{I}+\lambda_j^\nu\mathbb{P}_j^\nu)=(\mathbb{P}_j^\nu)_{\Re_j}+\lambda_j^\nu\mathbb{Q}_j^\nu+\left(\lambda_j^\nu\right)^2\mathbb{I}\det\mathbb{P}_j^\nu$$

with $\mathbb{Q}_j^\nu=[Q_{r,s}]_{p,q\in\{1,2,3\}}\in\mathcal{M}_{3,3}(\mathbb{R})$ and

$$\begin{array}{rcl}
Q_{11} &=& P_{11}\left(P_{33}+P_{22}\right)-P_{12}^2-P_{13}^2,\\
Q_{12} &=& P_{12}P_{33}-P_{13}P_{23},\\
Q_{13} &=& P_{13}P_{22}-P_{12}P_{23},\\
Q_{22} &=& P_{22}\left(P_{33}+P_{11}\right)-P_{12}^2-P_{23}^2,\\
Q_{23} &=& P_{23}P_{11}-P_{12}P_{13},\\
Q_{33} &=& P_{33}\left(P_{22}+P_{11}\right)-P_{13}^2-P_{23}^2.
\end{array}$$

As $\det\mathbb{P}_j^\nu=0$, we obtain :

$$(S_{\lambda_j^\nu}(\mathbb{P}_j^\nu))_{\Re_j^\nu}=\frac{1}{\det(\mathbb{I}+\lambda_j^\nu\mathbb{P}_j^\nu)}\left((\mathbb{P}_j^\nu)_{\Re_j^\nu}+\lambda_j^\nu\mathbb{Q}_j^\nu\right)$$

and

$$(S_{\lambda_j^\nu}(\mathbb{P}_j^\nu))_{\Re_{j+1}^\nu}\quad=\quad\tfrac{1}{\det(\mathbb{I}+\lambda_j^\nu\mathbb{P}_j^\nu)}\;\mathbb{R}_j^\nu\left((\mathbb{P}_j^\nu)_{\Re_j}+\lambda_j^\nu\mathbb{Q}_j^\nu\right)\,{}^t\mathbb{R}_j^\nu.$$

Now, we have to compute $T_{\mathbb{B}(\sigma_{j+1}^\nu),n(\sigma_{j+1}^\nu),\xi_j^\nu}\left(S_{\lambda_j^\nu}(\mathbb{P}_j^\nu)\right)_{\Re_{j+1}^\nu}$.

In local coordinates, we have $\mathbb{B}(\sigma_{j+1}^\nu)=\begin{bmatrix}\frac{1}{U_{j+1}^\nu}&0&0\\0&\frac{1}{V_{j+1}^\nu}&0\\0&0&0\end{bmatrix}_{\Re_{j+1}^\nu}$, $n(\sigma_{j+1}^\nu)_{\Re_{j+1}^\nu}=\begin{pmatrix}0\\0\\1\end{pmatrix}$ and $\left(\xi_{j+1}^\nu\right)_{\Re_{j+1}^\nu}=\begin{pmatrix}\xi_{j+1,1}^\nu\\\xi_{j+1,2}^\nu\\\xi_{j+1,3}^\nu\end{pmatrix}$. So, if we note $\mathbb{W}=\left[(w_{pq})_{p,q\in\{1,2,3\}}\right]=(S_{\lambda_j^\nu}(\mathbb{P}_j^\nu))_{\Re_{j+1}^\nu}$ and $\mathbb{T}=\left[(t_{pq})_{p,q\in\{1,2,3\}}\right]=\left(T_{\mathbb{B}(\sigma_{j+1}^\nu),n(\sigma_{j+1}^\nu),\xi_j^\nu}\left(S_{\lambda_j^\nu}(\mathbb{P}_j^\nu)\right)\right)_{\Re_{j+1}^\nu}$ then, $\forall x\in\mathbb{R}^3$

$$\begin{array}{c}
\left(T_{\mathbb{B}(\sigma_{j+1}^\nu),n(\sigma_{j+1}^\nu),\xi_j^\nu}\left(S_{\lambda_j^\nu}(\mathbb{P}_j^\nu)\right)\right)_{\Re_{j+1}^\nu}x\\
=\\
\begin{bmatrix}
w_{11}-2\frac{\left\langle\xi_j^\nu,n(\sigma_{j+1}^\nu)\right\rangle}{U_{j+1}^\nu} & w_{12} & w_{13}\\
w_{12} & w_{22}-2\frac{\left\langle\xi_j^\nu,n(\sigma_{j+1}^\nu)\right\rangle}{V_{j+1}^\nu} & w_{23}\\
w_{13} & w_{23} & w_{33}
\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\\
-2x_3\begin{bmatrix}
w_{13}+\frac{\left\langle\xi_j^\nu,\mathbf{u}_{j+1}^\nu\right\rangle}{U_{j+1}^\nu}\\
w_{23}+\frac{\left\langle\xi_j^\nu,\mathbf{v}_{j+1}^\nu\right\rangle}{V_{j+1}^\nu}\\
w_{33}
\end{bmatrix}\\
-2\left(\begin{bmatrix}
w_{13}+\frac{\left\langle\xi_j^\nu,\mathbf{u}_{j+1}^\nu\right\rangle}{U_{j+1}^\nu}\\
w_{23}+\frac{\xi_{j,2}^\nu}{V_{j+1}^\nu}\\
w_{33}
\end{bmatrix},\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)\begin{bmatrix}0\\0\\1\end{bmatrix}\\
+4\left(\begin{bmatrix}w_{13}\\w_{23}\\w_{33}\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right)x_3\begin{bmatrix}0\\0\\1\end{bmatrix}\\
-\frac{2}{\left\langle\xi_j^\nu,n(\sigma_{j+1}^\nu)\right\rangle}\left(\frac{\left\langle\xi_j^\nu,\mathbf{u}_{j+1}^\nu\right\rangle^2}{U_{j+1}^\nu}+\frac{\left\langle\xi_j^\nu,\mathbf{v}_{j+1}^\nu\right\rangle^2}{V_{j+1}^\nu}\right)x_3\begin{bmatrix}0\\0\\1\end{bmatrix}
\end{array}$$

thus

$$\begin{aligned}
t_{11} &= w_{11} - 2\frac{\left\langle \xi_j^\nu, n(\sigma_{j+1}^\nu)\right\rangle}{U_{j+1}^\nu},\\
t_{12} &= w_{12},\\
t_{13} &= -w_{13} - 2\frac{\left\langle \xi_j^\nu, \mathbf{u}_{j+1}^\nu\right\rangle}{U_{j+1}^\nu},\\
t_{22} &= w_{22} - 2\frac{\left\langle \xi_j^\nu, n(\sigma_{j+1}^\nu)\right\rangle}{V_{j+1}^\nu},\\
t_{23} &= -w_{23} - 2\frac{\left\langle \xi_j^\nu, \mathbf{v}_{j+1}^\nu\right\rangle}{V_{j+1}^\nu},\\
t_{33} &= w_{33} - \frac{2}{\left\langle \xi_j^\nu, n(\sigma_{j+1}^\nu)\right\rangle}\left(\frac{\left\langle \xi_j^\nu, \mathbf{u}_{j+1}^\nu\right\rangle^2}{U_{j+1}^\nu} + \frac{\left\langle \xi_j^\nu, \mathbf{v}_{j+1}^\nu\right\rangle^2}{V_{j+1}^\nu}\right).
\end{aligned}$$

Using $\left\langle \xi_j^\nu, n(\sigma_{j+1}^\nu)\right\rangle = \left(1-2\delta^\nu(\sigma_{j+1}^\nu)\right)\left\langle \xi_{j+1}^\nu, n(\sigma_{j+1}^\nu)\right\rangle$, we obtain :

$$\left(T_{\mathbb{B}(\sigma_{j+1}^\nu),n(\sigma_{j+1}^\nu),\xi_j^\nu}\left(S_{\lambda_j^\nu}(\mathbb{P}_j^\nu)\right)\right)_{|T\partial\Omega(\sigma_{j+1}^\nu)}$$
$$=$$
$$\left((S_{\lambda_j^\nu}(\mathbb{P}_j^\nu))_{\Re_{j+1}}\right)_{|T\partial\Omega(\sigma_{j+1}^\nu)} - 2\left(1-2\delta^\nu(\sigma_{j+1}^\nu)\right)\left\langle \xi_{j+1}^\nu, n(\sigma_{j+1}^\nu)\right\rangle \mathbb{B}(\sigma_{j+1}^\nu)_{|T\partial\Omega(\sigma_{j+1}^\nu)}$$

Finally, we have

$$\begin{aligned}
&\left((\mathbb{P}_{j+1}^\nu)_{\Re_{j+1}}\right)_{|T\partial\Omega(\sigma_{1+1}^\nu)}\\
&=\\
&\left((S_{\lambda_j^\nu}(\mathbb{P}_j^\nu))_{\Re_{j+1}}\right)_{|T\partial\Omega(\sigma_{j+1}^\nu)}\\
&+2\delta^\nu(\sigma_{j+1}^\nu)\left\langle \xi_{j+1}^\nu, n(\sigma_{j+1}^\nu)\right\rangle \mathbb{B}(\sigma_{j+1}^\nu)_{|T\partial\Omega(\sigma_{j+1}^\nu)}
\end{aligned} \tag{5.6}$$

- **Computation of** $\mathbb{M}_{j+1}^\nu$
  By definition,

$$\mathbb{M}_{j+1}^\nu = H^S_{\sigma_{j+1}^\nu,\sigma_{j+1}^\nu}\psi_l(x,\nu) - H^S_{\sigma_j^\nu,\sigma_{j+1}^\nu}\psi_l(x,\nu)\left[\mathbb{M}_j^\nu\right]^{-1}H^S_{\sigma_{j+1}^\nu,\sigma_j^\nu}\psi_l(x,\nu)$$

We first evaluate $H^S_{\sigma_{j+1}^\nu,\sigma_j^\nu}\psi_l(x;\nu)[\mathbb{M}_j^\nu]^{-1}H^S_{\sigma_j^\nu,\sigma_{j+1}^\nu}\psi_l(x;\nu)$. By construction of ψ_l, we have :

$$H^S_{\sigma_{j+1}^\nu,\sigma_j^\nu}\psi_l(x;\nu) = H^S_{\sigma_{j+1}^\nu,\sigma_j^\nu}\mid \sigma_{j+1}^\nu - \sigma_j^\nu \mid .$$

That's give in $\Re_j$

$$H^S_{\sigma_{j+1}^\nu,\sigma_j^\nu}\psi_l(x;\nu) = \frac{1}{\lambda_j^\nu}\mathbb{L}_j^\nu$$

where $\mathbb{L}_j^\nu = \left[L_{p,q}\right]_{p,q\in\{1,2\}}$ with

$$\begin{aligned}
L_{11} &= \left\langle \mathbf{u}_j^\nu, \xi_j^\nu\right\rangle\left\langle \mathbf{u}_{j+1}^\nu, \xi_j^\nu\right\rangle - \left\langle \mathbf{u}_j^\nu, \mathbf{u}_{j+1}^\nu\right\rangle,\\
L_{12} &= \left\langle \mathbf{v}_j^\nu, \xi_j^\nu\right\rangle\left\langle \mathbf{u}_{j+1}^\nu, \xi_j^\nu\right\rangle - \left\langle \mathbf{v}_j^\nu, \mathbf{u}_{j+1}^\nu\right\rangle,\\
L_{21} &= \left\langle \mathbf{u}_j^\nu, \xi_j^\nu\right\rangle\left\langle \mathbf{v}_{j+1}^\nu, \xi_j^\nu\right\rangle - \left\langle \mathbf{u}_j^\nu, \mathbf{v}_{j+1}^\nu\right\rangle,\\
L_{22} &= \left\langle \mathbf{v}_j^\nu, \xi_j^\nu\right\rangle\left\langle \mathbf{v}_{j+1}^\nu, \xi_j^\nu\right\rangle - \left\langle \mathbf{v}_j^\nu, \mathbf{v}_{j+1}^\nu\right\rangle.
\end{aligned}$$

We have by hypothesis $\mathbb{M}_j^\nu = \mathbb{P}^\nu_{j\,|T\partial\Omega(\sigma_j^\nu)} + \frac{1}{\lambda_j^\nu}\left(\mathbb{I} - (\xi_j^\nu\ {}^t\xi_j^\nu)\right)_{|T\partial\Omega(\sigma_j^\nu)}$ i.e. :

$$\begin{aligned}
\left(\mathbb{M}_j^\nu\right)_{\Re_j^\nu} &= \left[M_{p,q}\right]_{p,q\in\{1,2\}}\\
&= \begin{bmatrix} P_{11} & P_{12}\\ P_{12} & P_{22}\end{bmatrix} + \frac{1}{\lambda_j^\nu}\begin{bmatrix} 1-\left(\xi_{j,1}^\nu\right)^2 & -\xi_{j,1}^\nu\xi_{j,2}^\nu\\ -\xi_{j,1}^\nu\xi_{j,2}^\nu & 1-\left(\xi_{j,2}^\nu\right)^2\end{bmatrix}
\end{aligned}$$

By recurrence hypothesis $\mathbb{M}_j^\nu$ is regular, so we have :

$$H^S_{\sigma^\nu_{j+1},\sigma^\nu_j}\psi_l(x;\nu)[\mathbb{M}_j^\nu]^{-1}H^S_{\sigma^\nu_j,\sigma^\nu_{j+1}}\psi_l(x;\nu) = \frac{1}{\left(\lambda_j^\nu\right)^2}\mathbb{L}_j^\nu[\mathbb{M}_j^\nu]^{-1}\,{}^t\mathbb{L}_j^\nu.$$

Now, we evaluate $H^S_{\sigma^\nu_{j+1},\sigma^\nu_{j+1}}\psi_l(x;\nu)$ in $\Re_{j+1}$. We find

$$\begin{array}{c} H^S_{\sigma^\nu_{j+1},\sigma^\nu_{j+1}}\psi_l(x;\nu) \\ = \\ H^S_{\sigma^\nu_{j+1},\sigma^\nu_{j+1}}\left(\mid \sigma^\nu_{j+1}-\sigma^\nu_j \mid + \mid \sigma^\nu_{j+2}-\sigma^\nu_{j+1}\mid\right) \\ = \\ \frac{1}{\lambda_j^\nu}\left(\mathbb{I}-\xi_j^\nu\,{}^t\xi_j^\nu\right)_{|T\partial\Omega(\sigma^\nu_{j+1})} + \frac{1}{\lambda^\nu_{j+1}}\left(\mathbb{I}-\xi^\nu_{j+1}\,{}^t\xi^\nu_{j+1}\right)_{|T\partial\Omega(\sigma^\nu_{j+1})} \\ -2\delta^\nu(\sigma^\nu_{j+1})\left\langle n(\sigma^\nu_{j+1}),\xi_j^\nu\right\rangle\mathbb{B}(\sigma^\nu_{j+1})_{|T\partial\Omega(\sigma^\nu_{j+1})}. \end{array}$$

Finally, we obtain

$$\begin{array}{c} \mathbb{M}^\nu_{j+1} \\ = \\ \frac{1}{\lambda_j^\nu}\left(\mathbb{I}-\xi_j^\nu\,{}^t\xi_j^\nu\right)_{|T\partial\Omega(\sigma^\nu_{j+1})} + \frac{1}{\lambda^\nu_{j+1}}\left(\mathbb{I}-\xi^\nu_{j+1}\,{}^t\xi^\nu_{j+1}\right)_{|T\partial\Omega(\sigma^\nu_{j+1})} \\ -2\delta^\nu(\sigma_{j+1})\left\langle n(\sigma^\nu_{j+1}),\xi_j^\nu\right\rangle\mathbb{B}(\sigma_{j+1})_{|T\partial\Omega(\sigma^\nu_{j+1})} - \frac{1}{\left(\lambda_j^\nu\right)^2}\mathbb{L}_j^\nu[\mathbb{M}_j^\nu]^{-1}\,{}^t\mathbb{L}_j^\nu. \end{array} \tag{5.7}$$

- **Formula (5.2) at step $j+1$:**
We compute now $\mathbb{D}^\nu_{j+1} = \mathbb{P}^\nu_{j+1}{}_{|T\partial\Omega(\sigma^\nu_{j+1})} + \frac{1}{\lambda^\nu_{j+1}}\left(\mathbb{I}-(\xi_{j+1}\ {}^t\xi_{j+1})_{|T\partial\Omega(\sigma^\nu_{j+1})}\right) -$ $\mathbb{M}^\nu_{j+1}$. Combining formulas (5.6) et (5.7), we obtain

$$\begin{array}{c} \mathbb{D}^\nu_{j+1} \\ = \\ \left((S_{\lambda_j^\nu}(\mathbb{P}_j^\nu))_{\Re_{j+1}}\right)_{|T\partial\Omega(\sigma^\nu_{j+1})} + \frac{1}{\lambda_j^\nu}\left(\mathbb{I}-\xi_j^\nu\,{}^t\xi_j^\nu\right)_{|T\partial\Omega(\sigma^\nu_{j+1})} + \frac{1}{\left(\lambda_j^\nu\right)^2}\mathbb{L}_j^\nu[\mathbb{M}_j^\nu]^{-1}\,{}^t\mathbb{L}_j^\nu. \end{array}$$

We prove that $\mathbb{D}^\nu_{j+1}=0$ using relations $\mathbb{P}_j^\nu\xi_j^\nu=0$, $\mid \xi_j^\nu \mid = 1$ and ${}^t\mathbb{R}_j^\nu\mathbb{R}_j^\nu=\mathbb{I}$.
- **Formula (5.3) at step $j+1$:**
We remark that $\mathbb{P}^\nu_{j+1}\xi^\nu_{j+1}=0$. Then, we pose

$$\mathbb{H}^\nu_{j+1} = \begin{bmatrix} 1 & 0 & \xi^\nu_{j+1,1} \\ 0 & 1 & \xi^\nu_{j+1,2} \\ 0 & 0 & \xi^\nu_{j+1,3} \end{bmatrix}$$

and we obtain

$$\det\left(\mathbb{I}+\lambda^\nu_{j+1}\mathbb{P}^\nu_{j+1}\right) = \frac{1}{(\det\mathbb{H}^\nu_{j+1})^2}\det\left({}^t\mathbb{H}^\nu_{j+1}\left(\mathbb{I}+\lambda^\nu_{j+1}\mathbb{P}^\nu_{j+1}\right)\mathbb{H}^\nu_{j+1}\right).$$

So

$$\begin{array}{c} \left\langle\xi^\nu_{j+1},n(\sigma^\nu_{j+1})\right\rangle^2\det\left(\mathbb{I}+\lambda^\nu_{j+1}\mathbb{P}^\nu_{j+1}\right) \\ = \\ \det\begin{bmatrix} 1+\lambda^\nu_{j+1}\left\langle\mathbb{P}^\nu_{j+1}\mathbf{u}^\nu_{j+1},\mathbf{u}^\nu_{j+1}\right\rangle & \lambda^\nu_{j+1}\left\langle\mathbb{P}^\nu_{j+1}\mathbf{u}^\nu_{j+1},\mathbf{v}^\nu_{j+1}\right\rangle & \left\langle\xi^\nu_{j+1},\mathbf{u}^\nu_{j+1}\right\rangle \\ \lambda^\nu_{j+1}\left\langle\mathbb{P}^\nu_{j+1}\mathbf{u}^\nu_{j+1},\mathbf{v}^\nu_{j+1}\right\rangle & 1+\lambda^\nu_{j+1}\left\langle\mathbb{P}^\nu_{j+1}\mathbf{v}^\nu_{j+1},\mathbf{v}^\nu_{j+1}\right\rangle & \left\langle\xi^\nu_{j+1},\mathbf{v}^\nu_{j+1}\right\rangle \\ \left\langle\xi^\nu_{j+1},\mathbf{u}^\nu_{j+1}\right\rangle & \left\langle\xi^\nu_{j+1},\mathbf{v}^\nu_{j+1}\right\rangle & 1 \end{bmatrix}. \end{array}$$

As $\mid \xi^\nu_{j+1}\mid = 1$, we get, with formula (5.2) at step $j+1$:

$$\det\left(\mathbb{I}+\lambda^\nu_{j+1}\mathbb{P}^\nu_{j+1}\right) = \left(\frac{\lambda^\nu_{j+1}}{\left\langle\xi^\nu_{j+1},n(\sigma^\nu_{j+1})\right\rangle}\right)^2\det\mathbb{M}^\nu_{j+1}.$$

- **Formulas (5.4) and (5.5) at step $j+1$:**

  In fact, we proved that
  $$\forall\lambda>0 \quad \det(\mathbb{I}+\lambda\mathbb{P}^{\rho}_{j+1}) = \left(\frac{\lambda}{\langle \xi^{\rho}_{j+1}, n(\sigma^{\rho}_{j+1})\rangle}\right)^2 \det\mathbb{M}^{\rho}_{j+1}$$
  where $\rho=(\sigma^{\nu}_1,\cdots,\sigma^{\nu}_{j+1})\in\mathcal{C}_{j+1}(\lambda\xi^{\nu}_{j+1})$. Moreover, we have :
  $$\mathbb{P}^{\rho}_{j+1}=\mathbb{P}^{\nu}_{j+1},$$
  for $\lambda=\lambda^{\nu}_{j+1}$,
  $$\mathbb{M}^{\rho}_{j+1}=\mathbb{M}^{\nu}_{j+1}$$
  and $\forall\lambda>0$
  $$\det(\mathbb{I}+\lambda\mathbb{P}^{\rho}_{j+1})>0$$
  because $\mathbb{P}^{\rho}_{j+1}$ is a positive matrix. That's give
  $$\forall\lambda>0, \quad \det\mathbb{M}^{\rho}_{j+1}\geq 0.$$
  This quantity is positive for λ in a neighborhood of zero, then by continuity we have :
  $$\forall\lambda>0 \quad \operatorname{tr}\mathbb{M}^{\rho}_{j+1}>0$$
  with
  $$\operatorname{tr}\mathbb{M}^{\rho}_{j+1}=P_{11}+P_{22}+\frac{1+\langle \xi^{\nu}_{j+1}, n(\sigma^{\nu}_{j+1})\rangle^2}{\lambda}.$$
  Thus, we obtain formula (5.4) to step $j+1$.
  We have also showed that $\mathbb{M}^{\nu}_{j+1}$ is regular.

That's close the proof of Lemma 2□

5.3. **Lemma of transmission.**

Lemma 3 (of transmission)**.** *Let $x\in\Omega\setminus\mathcal{T}$ and $\nu=(\sigma^{\nu}_1,\cdots,\sigma^{\nu}_l)\in\mathcal{C}_l(x)$*

(1) *if exists $t\in\mathbb{R}^+_*$ such that*
$$\sigma=\sigma^{\nu}_1-t\xi \ \in\partial\Omega \text{ and } \xi.n(\sigma)<0$$
then
$$\mu=(\sigma,\sigma^{\nu}_1,\cdots,\sigma^{\nu}_l)\in\mathcal{C}_{l+1}(x)$$
(2) *if exists $j\in\{1,\cdots,l-1\}$ such that*
$$\sigma\in\left\{]\sigma^{\nu}_j;\sigma^{\nu}_{j+1}[\cap\partial\Omega\right\} \text{ and } (\sigma-\sigma^{\nu}_j).n(\sigma)<0$$
then
$$\mu=(\sigma^{\nu}_1,\cdots,\sigma^{\nu}_j,\sigma,\sigma^{\nu}_{j+1},\cdots,\sigma^{\nu}_l)\in\mathcal{C}_{l+1}(x)$$

Noting $\mu=(\sigma^{\mu}_1,\cdots,\sigma^{\mu}_{l+1})$, we obtain in both cases
$$\begin{aligned}&(-1)^l\frac{e^{-ik\psi_l(x;\nu)}}{\left|\prod\limits_{j=1}^{l}\det\left(\mathbb{I}+|\sigma^{\nu}_{j+1}-\sigma^{\nu}_j|\mathbb{P}^{\nu}_j\right)\right|^{1/2}}\\ &+(-1)^{l+1}\frac{e^{-ik\psi_{l+1}(x;\mu)}}{\left|\prod\limits_{j=1}^{l+1}\det\left(\mathbb{I}+|\sigma^{\mu}_{j+1}-\sigma^{\mu}_j|\mathbb{P}^{\mu}_j\right)\right|^{1/2}}=0\end{aligned} \tag{5.8}$$

Proof of Lemma 3 :

In both case, using the strict convexity of compacts $(K_j)_{j\in\{1,\cdots,N\}}$, we obtain $\mu\in\mathcal{C}_{l+1}(x)$.

Remark 3. *In two cases, μ realize a transmission condition in σ, and thus*
$$\psi_l(x;\nu)=\psi_{l+1}(x;\mu)$$

Now, we prove the formula (5.8), in both case. In fact, we only have to prove that

$$\left|\prod_{j=1}^{l}\det\left(\mathbb{I}+\mid\sigma_{j+1}^{\nu}-\sigma_{j}^{\nu}\mid\mathbb{P}_{j}^{\nu}\right)\right|^{1/2}=\left|\left(\prod_{j=1}^{l+1}\det\left(\mathbb{I}+\mid\sigma_{j+1}^{\mu}-\sigma_{j}^{\mu}\mid\mathbb{P}_{j}^{\mu}\right)\right)\right|^{1/2}$$

Here $\sigma_{l+1}^{\nu}=\sigma_{l+2}^{\mu}=x$.
In the first case, the proof is immediate.
Under the hypothesis of the second case, we have

$$\forall j\in\{1,\cdots,j\}\ \ \mathbb{P}_{j}^{\nu}=\mathbb{P}_{j}^{\mu}$$

and, μ realize a transmission condition in σ, hence

$$\mathbb{P}_{j+1}^{\mu}=S_{|\sigma_{j}^{\nu}-\sigma|}(\mathbb{P}_{j}^{\nu})=\mathbb{P}_{j}^{\nu}\left(\mathbb{I}+\mid\sigma_{j}^{\nu}-\sigma\mid\mathbb{P}_{j}^{\nu}\right)^{-1}.$$

Thus, we obtain

$$\left(\mathbb{I}+\mid\sigma_{j+1}^{\nu}-\sigma\mid\mathbb{P}_{j+1}^{\mu}\right)\left(\mathbb{I}+\mid\sigma_{j}^{\nu}-\sigma\mid\mathbb{P}_{j}^{\mu}\right)=\mathbb{I}+\mid\sigma_{j+1}^{\nu}-\sigma_{j}^{\nu}\mid\mathbb{P}_{j}^{\nu}.$$

Taking determinant of previous formula, we get

$$\det\left(\mathbb{I}+\mid\sigma_{j+1}^{\nu}-\sigma\mid\mathbb{P}_{j+1}^{\mu}\right)\det\left(\mathbb{I}+\mid\sigma_{j}^{\nu}-\sigma\mid\mathbb{P}_{j}^{\mu}\right)=\det\left(\mathbb{I}+\mid\sigma_{j+1}^{\nu}-\sigma_{j}^{\nu}\mid\mathbb{P}_{j}^{\nu}\right).$$

Moreover, we have

$$S_{|\sigma_{j+2}^{\nu}-\sigma_{j+1}^{\mu}|}(\mathbb{P}_{j+1}^{\mu})=S_{|\sigma_{j+1}^{\nu}-\sigma|}\left(S_{|\sigma_{j}^{\nu}-\sigma|}(\mathbb{P}_{j}^{\nu})\right)$$

As μ realize a transmission condition in σ we obtain

$$\mid\sigma_{j+1}^{\nu}-\sigma\mid+\mid\sigma_{j+1}^{\nu}-\sigma\mid=\mid\sigma_{j+2}^{\nu}-\sigma_{j+1}^{\mu}\mid$$

thus

$$S_{|\sigma_{j+2}^{\nu}-\sigma_{j+1}^{\mu}|}(\mathbb{P}_{j+1}^{\mu})=S_{|\sigma_{j+1}^{\nu}-\sigma_{j}^{\nu}|}(\mathbb{P}_{j}^{\nu}).$$

Then, we have

$$\forall i\in\{j+1,\cdots,l\}\ \ \mathbb{P}_{i}^{\nu}=\mathbb{P}_{i+1}^{\mu}$$

That's close proof of Lemma 3.□

6. Proof of Theorem 1

To proof this theorem, we apply stationary phase technics to the formula (4.1) and compare the result to the geometrical optic approximation.

6.1. **Stationary phase lemma.** Due to (5.5) ($\mathbb{M}_{j}^{\nu}$ regular) we can apply the iterative stationary phase lemma to $u_{l}^{D}(x)$ (see [Cuv94]) : $\forall x\in\Omega\setminus\mathcal{T}$

$$u_{l}^{D}(x)-u_{l-1}^{D}(x)=\left(\tfrac{ik}{4\pi}\right)^{l}\left(\tfrac{2\pi}{k}\right)^{l}\sum_{\nu\in\mathcal{C}_{l}(x)}\frac{e^{i\frac{\pi}{4}\sum_{j=1}^{l}\operatorname{sgn}\mathbb{M}_{j}^{\nu}}}{\left|\prod_{j=1}^{l}\det\mathbb{M}_{j}^{\nu}\right|^{1/2}}\mathcal{D}_{l}(\nu)\frac{e^{-ik\psi_{l}(x;\nu)}}{|x-\sigma_{l}^{\nu}|}+O(\tfrac{1}{k}).\tag{6.1}$$

Thus, using definition of $\mathcal{D}_{l}(\nu)$ with $\nu\in\mathcal{C}_{l}(x)$, we obtain

$$\mathcal{D}_{l}(\nu)=\left(2^{l}\right)|\langle\xi,n(\sigma_{1}^{\nu})\rangle|\prod_{j=1}^{l-1}\frac{\left|\left\langle\frac{\sigma_{j+1}^{\nu}-\sigma_{j}^{\nu}}{|\sigma_{j+1}^{\nu}-\sigma_{j}^{\nu}|},n(\sigma_{j+1}^{\nu})\right\rangle\right|}{\mid\sigma_{j+1}^{\nu}-\sigma_{j}^{\nu}\mid}$$

Due to formula 5.3 and 5.4 (lemma 2), we have

$$\sum_{j=1}^{l}\operatorname{sgn}\mathbb{M}_{j}^{\nu}=2^{l}$$

and

$$\left|\prod_{j=1}^{l} \det \mathbb{M}_j^\nu\right|^{1/2} = \prod_{j=1}^{l} \frac{|\langle \xi_j^\nu, n(\sigma_j^\nu)\rangle|}{\lambda_j^\nu} \left|\det(\mathbb{I} + \lambda_j^\nu \mathbb{P}_j^\nu)\right|^{1/2}$$
$$= \prod_{j=1}^{l} \frac{\left|\left\langle \frac{\sigma_{j+1}^\nu - \sigma_j^\nu}{|\sigma_{j+1}^\nu - \sigma_j^\nu|}, n(\sigma_j^\nu)\right\rangle\right|}{|\sigma_{j+1}^\nu - \sigma_j^\nu|} \left|\det(\mathbb{I} + |\sigma_{j+1}^\nu - \sigma_j^\nu|\, \mathbb{P}_j^\nu)\right|^{1/2}$$

with $\sigma_{l+1}^\nu = x$. Taking into account that, for $\nu \in \mathcal{C}_l(x)$, we have

$$\left|\left\langle \frac{\sigma_2^\nu - \sigma_1^\nu}{|\sigma_2^\nu - \sigma_1^\nu|}, n(\sigma_1^\nu)\right\rangle\right| = |\langle \xi, n(\sigma_1^\nu)\rangle|,$$
$$\left|\left\langle \frac{\sigma_{j+1}^\nu - \sigma_j^\nu}{|\sigma_{j+1}^\nu - \sigma_j^\nu|}, n(\sigma_j^\nu)\right\rangle\right| = \left|\left\langle \frac{\sigma_j^\nu - \sigma_{j-1}^\nu}{|\sigma_j^\nu - \sigma_{j-1}^\nu|}, n(\sigma_j^\nu)\right\rangle\right|, \ j = 2, \ldots, l,$$

we find that

$$u_l^D(x) - u_{l-1}^D(x) = (-1)^l \sum_{\nu=(\sigma_1^\nu,\cdots,\sigma_l^\nu)\in\mathcal{C}_l(x)} \frac{e^{-ik\psi_l(x;\nu)}}{\left|\left(\prod_{j=1}^{l-1} \det\left(\mathbb{I}+|\sigma_{j+1}^\nu-\sigma_j^\nu|\mathbb{P}_j^\nu\right)\right) \det\left(\mathbb{I}+|x-\sigma_l^\nu|\mathbb{P}_l^\nu\right)\right|^{1/2}} + O(\tfrac{1}{k}) \tag{6.2}$$

6.2. **Comparison with geometrical optic approximation.** To compare the previous formula with geometrical optic approximation given by formula (3.1), we have to study the contributions of the sets $\mathcal{C}_l(x)$ and $\mathcal{R}_l(x)$. We clearly have $\mathcal{R}_l(x) \subset \mathcal{C}_l(x)$. Using lemma 3 we obtain

Remark 4. *The contributions of points in $\mathcal{C}(x) \setminus \mathcal{R}(x)$ cancel each other.*

Let $x \in \Omega \setminus \mathcal{T}$. Suppose that $\mathcal{C}_l(x) = \emptyset$ for $l > n$, we conclude the proof of Theorem 1 using the following remark

Remark 5. *The only components of $\mathcal{C}(x)$ having a real contribution are :*

- *all $\nu = (\sigma_1^\nu) \in \mathcal{C}_1(x)$ coming through x such that ν realize a transmission condition in σ_1^ν and*
$$\forall t > 0 \quad \sigma_1^\nu - t\xi \in \Omega$$
- *all $\nu = (\sigma_1^\nu, \cdots, \sigma_j^\nu) \in \mathcal{R}_j(x)$ coming through x ($j \le n$).*

That's close proof of Theorem 1.□

7. Proof of Theorem 2

To proof this theorem, we apply stationary phase technics to the formula (4.4) and compare the result to the geometrical optic approximation (3.2).

Due to (5.5) ($\mathbb{M}_j^\nu$ regular) we can apply the iterative stationary phase lemma to $u_l^N(x)$(see [Cuv94]),we obtain $\forall x \in \Omega \setminus \mathcal{T}$

$$u_l^N(x) - u_{l-1}^N(x) = \left(\tfrac{ik}{4\pi}\right)^l \left(\tfrac{2\pi}{k}\right)^l \sum_{\nu\in\mathcal{C}_l(x)} \frac{e^{i\frac{\pi}{4}\sum_{j=1}^{l} \operatorname{sgn} \mathbb{M}_j^\nu}}{\left|\prod_{j=1}^{l} \det \mathbb{M}_j^\nu\right|^{1/2}} \mathcal{N}_l(\nu) \left\langle \frac{x-\sigma_l^\nu}{|x-\sigma_l^\nu|}, n(\sigma_l^\nu)\right\rangle \frac{e^{-ik\psi_l(x;\nu)}}{|x-\sigma_l^\nu|} + O(\tfrac{1}{k}) \tag{7.1}$$

Thus, we use (5.3), (5.4) and the definition of $\mathcal{N}_l$ to get :

$$u_l^N(x) - u_{l-1}^N(x) = \sum_{\nu=(\sigma_1^\nu,\cdots,\sigma_l^\nu)\in\mathcal{C}_l(x)} \frac{e^{-ik\psi_l(x;\nu)}}{\left|\left(\prod_{j=1}^{l-1}\det\left(\mathbb{I}+|\sigma_{j+1}^\nu-\sigma_j^\nu|\mathbb{P}_j^\nu\right)\right)\det\left(\mathbb{I}+|x-\sigma_l^\nu|\mathbb{P}_l^\nu\right)\right|^{1/2}} + O\left(\frac{1}{k}\right) \tag{7.2}$$

To compare the previous formula with geometrical optic approximation given by formula (3.2), we use previous results from the proof of theorem 1.

8. Conclusion

We have proved the validity of the iterative Kirchhoff formulas (4.1) and (4.4) at high frequency. Numerical results for Dirichlet problem (D) can be founded in [Cuv94].

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